# Flexible pavement response models for assessing dynamic axle loads

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Several different dynamic road response models are compared and it is concluded that simple models, such as a beam on a Winkler foundation, can reproduce the characteristics of more complex models, such as a stack of elastic plates. Two different methods of using harmonic response models to reproduce road responses to moving random loads are compared, the convolution and influence function methods. It is concluded that the simpler of the two methods is sufficiently accurate for predicting road damage due to dynamic tyre forces.

## 1. INTRODUCTION

There are many different dynamic road models in general use, ranging from an elastic beam on a Winkler foundation (Cebon 1987) to more complex non-linear (Semenov 1976), anisotropic (Anon) or dynamic layered systems (Kausel and Peek 1982). The more complex the model, the more time (and effort) is required to solve a particular problem and this may prove unwarranted as the increased detail of the model may not be relevant to the problem addressed.

This paper examines the qualitative and quantitative results that may be derived from three linear, dynamic road models and presents formulae for calculating pertinent strains and displacements. A general convolution method is also presented for calculating the response of a linear system to moving dynamic loads from harmonic or impulse response functions. It is found that this method requires considerable computer time and therefore a simplification of the general method, the influence function method, is also presented. The paper evaluates the suitability of these two approaches for predicting road damage produced by vehicles, and quantifies the worst-case errors that may be introduced by the influence function method.

## 2. DYNAMIC ROAD MODELS

### 2.1 Convolution and influence function methods

The well-known convolution integral, used to give the output of a linear system to an arbitrarily varying input, may be extended to cater for moving loads. This formulation relies on knowledge of the road's impulse response functions which can either be measured on an instrumented road (Hardy and Cebon 1989) or calculated from the harmonic responses of one of the road models that follow. This approach assumes that the road response is linear and isotropic. This method has been compared favourably with extensive field measurements (Hardy 1990; Hardy and Cebon 1989).

It is possible to simplify the convolution method if the impulse responses of the road die away quickly with respect to the rate of change of the tyre loads. The simplified formulation predicts vehicle speed effects accurately but with the loss of a full dynamic calculation. The saving in computer time is, however, approximately a factor of 30–50.

### 2.2 Comparison of models

#### 2.2.1 Beam on a Winkler foundation

A beam on a damped elastic (Winkler) foundation is probably the simplest, useful road model, making it very attractive when developing a conceptual understanding of the way in which roads support loads. It has been compared with more complex dynamic layered models and found to behave qualitatively in the same way, showing similar responses to changes in material stiffnesses and densities (Hardy 1990).

The shortcomings of the beam model are that:

(1) The stiffness and damping parameters describing the subgrade do not directly relate to the material properties of the pavement.

(2) Only the stiff surface asphalt layers are modelled realistically and therefore the

behaviour and stress/strain distribution through the subgrade cannot be calculated.

(3) The stress/strain distribution through the surface layer is prescribed by the Euler Beam model and is not reliable calculating permanent deformation (rutting) in the asphalt layers.

(4) the model is restricted to a single space dimension and therefore cannot predict transverse strains. The peak transverse strain under a single load is equal to the peak longitudinal strain but this is not the case under multiple loads. This may cause errors in fatigue calculations.

#### 2.2.2 Plate on a Winkler foundation

The main advantage of the plate model over the beam is that transverse strains can be calculated. As the plate also has similar two-dimensional geometry as a road it is expected to yield more realistic results. The formulae involved are, however, much more complex than those for a beam. The first three limitations for the beam still apply, however, because the assumptions in simple plate theory are very similar to the assumptions in Euler Beam theory.

#### 2.2.3 Layered elastic half-space

This model uses an approach similar to the finite-element method to model the dynamic response of a stack of plates overlying a rigid foundation. Each element consists of an infinitely large, thin layer which is modelled as a plate but also with compliance through its thickness. It may therefore be used as a more refined model of a road structure than a single plate model and will yield any components of stress or strain that are be required. The parameters used in the model are the material properties (density, stiffness, damping, Poisson's ratio) and the layer thicknesses, so that the effects of varying materials can be investigated in all layers. This infinite-element approach requires substantial amounts of computer time and memory making it relatively unattractive for simple studies.

## 3. FORMULATIONS OF THE ROAD MODELS

### 3.1 The Convolution Method

The response of a linear system to a time-varying input is given by the convolution integral (Newland 1985):

$$y(t) = \int_{-\infty}^{\infty} h(t-\tau)\, f(\tau)\, d\tau \qquad (1)$$

where $y(t)$ is the response at time t,
$f(\tau)$ is the input force at time $\tau$,
and $h(t)$ is the response at time t to a unit impulse at time $t = 0$.

If the input is moving in a straight line at constant speed v with respect to the system and the response is measured at a point with position x then the convolution integral in equation (1) becomes:

$$y(x, t) = \int_{-\infty}^{\infty} h\,(x - v\tau,\, t-\tau)\, f(\tau)\, d\tau. \qquad (2)$$

where $y(x,t)$ is the response at position x at time t,
$h(x,t)$ is the response at position x, and time t, to a unit impulse at the origin at time $t=0$.

This equation is similar to that derived by Cebon (Cebon 1988b), where it was used to find the response of a continuously supported beam to arbitrary, moving excitation.

The simple convolution integral, equation (1), is often solved in the frequency domain using Fourier transforms because the transform of the integral reduces to a simple multiplication (Newland 1985).

The same technique may be used to simplify the moving convolution integral, equation (2), but it is necessary to take Fourier transforms with respect to both time and space variables as follows:

$$\tilde{y}(\xi, \omega)$$

$$= \left(\frac{1}{2\pi}\right)^2 \iint_{-\infty}^{\infty} y\,(x, t)\, e^{-i\omega t}\, e^{-i\xi x}\, dt\; dx$$

$$= \left(\frac{1}{2\pi}\right)^2 \iiint_{-\infty}^{\infty} h(x - v\tau, t - \tau) f(\tau)\, e^{-i\omega t} e^{-i\xi x}\, d\tau\, dt\, dx$$

$$= 2\pi \tilde{h}(\xi, \omega)\; \tilde{f}(\omega + v\xi) \qquad (3)$$

where $\omega$ is the angular frequency of loading, corresponding to the time t,
$\xi$ is the wave number, corresponding to the distance x,
and tilde, $\tilde{}$, indicates a transformed function.

This formula is particularly useful for ascertaining the relative importance of speed and frequency in road responses.

### 3.2 Influence Function Method

A change of variable in equation 2 gives the response as:

$$y(x, t) = \int_{-\infty}^{\infty} h(x - v(t-\tau), \tau)\, f(t-\tau)\, d\tau. \quad (4)$$

This may be simplified if f(t) changes much more slowly than the impulse response decays. In this case f(t-τ) may be considered constant over the integral and equation (4) reduces to:

$$y(x, t) = I(v, x-vt)\, f(t)$$

$$\text{where } I(v,x) = \int_{0}^{\infty} h(x+v\tau, \tau)\, d\tau. \quad (5)$$

I(v, x) is the 'influence function' at speed v and position x. The consequences of this simplification are discussed and quantified later.

### 3.3 Beam Equations

The motion of an infinite beam resting on a Winkler foundation and excited by an harmonic point load at the origin is described by the equation (Fryba 1972):

$$E^*I\frac{d^4w}{dx^4} + \left(k^* - \mu\omega^2\right)w = P\delta(x) \quad (6)$$

where $E^*I$ is the complex flexural rigidity of the beam,
w is the vertical beam displacement at a distance x from the origin,
k* is the complex support stiffness,
μ is the the mass per unit length of the beam
and P is the applied force.

The general solution of the fourth order differential equation is the weighted sum of four exponential functions and may be written as:

$$w = A_0e^{\beta x} + A_1e^{-\beta x} + A_2e^{i\beta x} + A_3e^{-i\beta x} \quad (7)$$

where $\beta^4 = (\mu\omega 2-k^*)/(E^*I)$ and β is taken with argument in the range -π/2 to 0.

This can be solved in either the damped case using boundary conditions of zero displacement and slope at large distances or in the undamped case requiring propagation of waves solely away from the origin. Both cases also require the zero slope and shear force compatibility at the origin,

$$\left.\frac{d^3w}{dx^3}\right|_{x=0} = \frac{-P}{2E^*I}. \quad (8)$$

The solution may be written as:

$$w = \frac{P}{4E^*\beta^3}\left(e^{-\beta|x|} - ie^{-i\beta|x|}\right). \quad (9)$$

The strain at the bottom of the beam is then given by:

$$\varepsilon_1 = \frac{h}{2}\frac{d^2w}{dx^2} = \frac{hP}{8E^*I\beta}\left(e^{-\beta|x|} + ie^{-i\beta|x|}\right) \quad (10)$$

where h is the thickness of the beam.

### 3.4 Plate Equations

The motion of an infinite plate resting on a Winkler foundation and excited by an harmonic point load at the origin is described by the equation (Fryba 1972):

$$D^*\nabla^4 w + \left(k^* - \mu\omega^2\right)w = \frac{P\delta(r)}{2\pi r} \quad (11)$$

where $D^*$ is the complex plate bending stiffness which is related to the complex Young's Modulus, $E^*$, Poisson's ratio, ν, and plate thickness, h, by $D^*=E^*h^3/(12(1-\nu^2))$,
$\nabla^4 w$ is the differential operator $\nabla^2$ acting twice on w to give: $\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right)\right)$,
w is the vertical displacement of the plate at a distance r from the origin,
k* is the complex support stiffness,
μ is the the mass per unit area of the plate
and P is the applied force.

The general solution of the fourth order differential equation is the weighted sum of four (Modified) Bessel functions and may be written as:

$$w = B_0J_0(\alpha r) + B_1Y_0(\alpha r) + B_2I_0(\alpha r) + B_3K_0(\alpha r) \quad (12)$$

where $\alpha^4 = (m\omega 2-k^*)/(D^*)$ and α is taken with argument in the range -π/2 to 0.

This may be solved, in either the damped or undamped cases, using boundary conditions of zero displacement and slope at large radii, or propagation of waves solely away from the origin, along with the zero slope condition and shear force compatibility at the origin,

$$r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left\{r\frac{dw}{dr}\right\}\right]\Bigg|_{r=0} = \frac{P}{2\pi D^*}. \tag{13}$$

The solution may be written most succinctly as a Thomson (Kelvin) function, (kei):

$$\begin{aligned} w &= \frac{-P}{8\alpha^2 D^*}\left[iJ_0(\alpha r) + Y_0(\alpha r) + \frac{2}{\pi}K_0(\alpha r)\right] \\ &= \frac{-Pi}{8\alpha^2 D^*}\,\text{kei}\left(\alpha r e^{i\pi/4}\right). \end{aligned} \tag{14}$$

Again, geometrical arguments give the longitudinal and transverse strains as:

$$\begin{aligned} \varepsilon_l &= \frac{h}{2}\frac{d^2w}{dr^2} \\ &= \frac{hP}{16D^*}\left(\text{ker}\left(\alpha r e^{i\pi/4}\right)\right. \\ &\quad \left. - \frac{e^{-i\pi/4}}{\sqrt{2}\,\alpha r}\left(\text{kei}_1\left(\alpha r e^{i\pi/4}\right) - \text{ker}_1\left(\alpha r e^{i\pi/4}\right)\right)\right) \end{aligned}$$

and

$$\begin{aligned} \varepsilon_t &= \frac{h}{2r}\frac{dw}{dr} \\ &= \frac{hPe^{-i\pi/4}}{16\sqrt{2}\,\alpha r D^*}\left(\text{kei}_1\left(\alpha r e^{i\pi/4}\right) - \text{ker}_1\left(\alpha r e^{i\pi/4}\right)\right). \end{aligned} \tag{15}$$

where $\varepsilon_l$ is the longitudinal strain,
$\varepsilon_t$ is the transverse strain,
ker, $\text{kei}_1$, $\text{kei}_1$ are Thomson functions
and h is the plate thickness.

## 3.5 A Layered Half-Space

For complete details of the method used to predict harmonic responses in layered media, reference should be made to the work of Kausel *et al.* (Kausel and Peek 1982; Kausel and Roësset 1981). An overview of a simplified method is given here for the sake of completeness. The simplifications are a consequence of considering only vertical surface loads with a circular, uniform pressure distribution. These loads cause displacements and stresses that may be defined uniquely by two components, the vertical and radial, as they generate no rotation about the axis of the load.

The axially symmetric stresses and displacements produced at layer interfaces by a vertical surface disc load may be transformed into the wave-number domain by use of the Hankel Transform. This transform breaks the loads and displacements at each layer in the road structure into their component Hankel functions (in the same way that the Fourier transform breaks a signal into its component sinusoids). The wave-number identifies each component of the Hankel Transform in the same way as angular frequency defines the Fourier component.

When transformed in this way the dynamic equilibrium of a single layer may be written as a matrix equation with a *stiffness matrix* relating transformed displacements to transformed loads. This term is used by Kausel to describe the matrix relating dynamic displacements to dynamic forces. It therefore includes a term which is dependent on the mass of the layer. For thin layers a linear interpolation function can be used for the vertical variation of displacements within the layer and in this case the *stiffness matrix* for a given layer contains simple algebraic expressions involving the geometric and material properties of that layer. If there is no slippage between layers the matrices for each layer of a multi-layered structure can be overlapped to give a global stiffness matrix because the internal stresses between layers are equal and opposite and the displacements of the bottom of each layer are the same as the displacements of the top of the layer below it.

In order to find the displacements due to the applied load it is necessary to invert the stiffness matrix. Kausel showed that the inverse matrix may be represented as the eigenvectors/values of an associated eigenvalue problem.

Once the elements of the inverse matrix are known it is possible to perform the inverse Hankel Transform to retrieve the desired displacements at each interface. The strains can also be derived from these general equations.

The vertical strain is constant in any given (thin) layer because of the linear interpolation function for displacements assumed above. In order to find the vertical strain in a layer it is only necessary to find the difference in the displacements of its two faces.

The radial (longitudinal) strain is given by the derivative of the radial displacement with radial distance:

$$\varepsilon_l = \frac{\partial u_r}{\partial r} \tag{16}$$

and the tangential (transverse) strain is given by:

$$\varepsilon_t = \frac{u_r}{r}. \tag{17}$$

The radial and tangential strains are principal strains because of the symmetry of loading and therefore the strain at any angle on an interface may be deduced from them by use of Mohr's circles. The vertical strain is not generally a principal strain for this loading case.

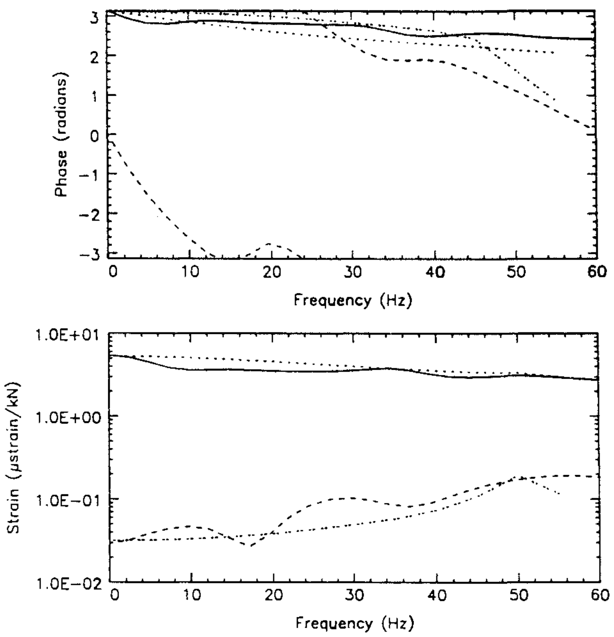



Figure 1 Measured and Simulated Frequency Response Functions

0m: Measured ———, Simulated ........,
1.8m: Measured – – – –., Simulated -- -- -- --

## 4 COMPARISON OF HARMONIC RESPONSES

A set of transverse strain impulse responses were measured on an instrumented road (Hardy and Cebon 1989). Fourier transforms were used to convert these measured impulse responses into harmonic response functions (Newland 1985). A set of parameters were chosen for use in the layered half-space model so that it would simulate these measured responses. The parameters chosen for the model are given in Table 1. The damping parameter, $\zeta$, is used to define the complex Young's Modulus and introduces viscous damping into the system. The complex modulus is given by $E^*=E_0(1 + i\omega\zeta)$. Figure 1 shows the measured and predicted frequency response functions with the load applied directly over the response position and also with the load applied at a distance of 1.8m from it. The comparison of the responses indicates that the layered model may be used to simulate the dynamic behaviour of the road accurately.

It is interesting to note that the depth of the subgrade (clay and hoggin layers) does not affect the distribution of strains through the pavement layers. Figure 2 shows the calculated static (i.e. 0Hz) strain as a function of depth directly under the load with subgrade depths of 1.6m, as shown in table 1, and 5.6m. The figure indicates that it is not necessary to model the subgrade precisely if only the strains in the surface layers are required. The interface between

| Material | Layer Depth (mm) | Modulus $E_0$ (MPa) | Damping $\zeta$ (sec/rad) | Poisson's Ratio | Density ($kg/m^3$) |
|---|---|---|---|---|---|
| Hot Rolled Asphalt | 50 | 3000 | $5x10^{-3}$ | 0.35 | 1000 |
| Dense Bituminous Macadam | 150 | 3000 | $5x10^{-3}$ | 0.35 | 1000 |
| Crushed Rock | 300 | 140 | $5x10^{-3}$ | 0.40 | 1500 |
| Clay | 600 | 140 | $1x10^{-4}$ | 0.45 | 1000 |
| Hoggin | 1000 | 140 | $1x10^{-4}$ | 0.45 | 1000 |

Table 1 Parameters for the layered half-space simulation.

the asphalt and aggregate layers is clearly visible at a depth of 0.2m and the almost linear distribution of strain through the surface layers make it attractive to model thick surface layers as a plates.

A comparison between the static deflection bowls for the layered half-space, a plate on a Winkler Foundation and a beam on a Winkler foundation is shown in figure 3. Figure 4 shows the variation in surface displacement directly under the load as a function of applied loading frequency for each of the models.

The material parameters for the plate and beam were chosen to be the same as the asphalt layers in the half-space. The width of the beam and the stiffness and damping of the Winkler foundations were then chosen to simulate the half-space data.

The effective stiffness of the support under a *rigid* plate (i.e. under plain-strain conditions) can be calculated from elasticity theory and is given by $k_0^*=E^*(1-v)/(h(1-v-2v^2))$ where h is the depth of the asphaltic layers. A value of $0.77k_0^*$ was found to give a good fit between the plate and half-space models. The plate is not rigid so the effective

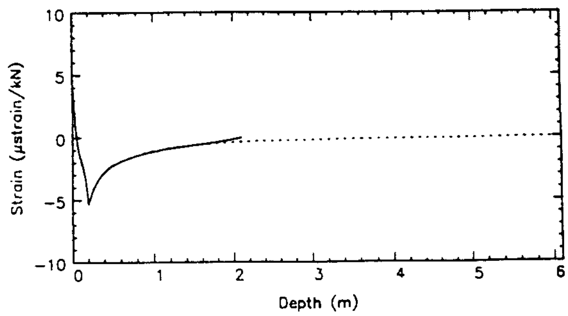



Figure 2 The Static Distribution of Horizontal Strain under the Load.

Subgrade depth 1.6m ———,
Subgrade depth 5.6m ........,

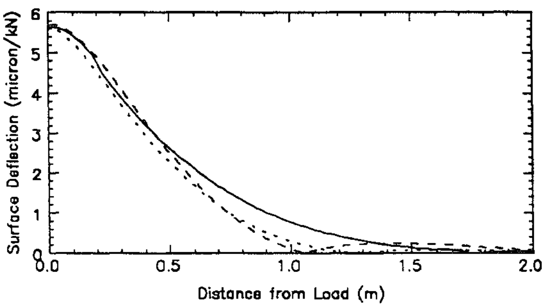
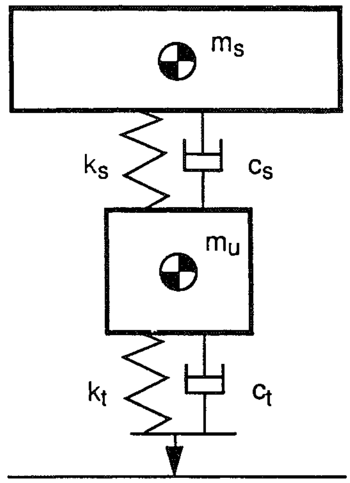



Figure 3 The Static Deflection Bowls.
Layered Half-Space ————,
Plate ·········
Beam – – – – ·




$m_s$ = 4450kg $m_u$ = 550kg
$c_s$ = 15kNs/m $c_t$ = 2kNs/m
$k_s$ = 1000kN/m $k_t$ = 1750kN/m

Figure 5 Quarter Car Vehicle Model.

foundation stiffness is reduced. The beam was chosen to be 1m wide (approximately half the size of the deflection bowl) and the required support stiffness was found to be $0.71k_0^*$. These figures illustrate that the beam and plate models can, under a restricted set of conditions, provide a useful model of a road.

## 5. COMPARISON OF CONVOLUTION AND INFLUENCE FUNCTION CALCULATIONS.

To compare the damage caused by dynamic tyre forces of heavy vehicles it is necessary to use either a convolution or influence function calculation. It is therefore important to know the magnitude of the errors in theoretical damage associated with the simplification of the influence function calculation.

To compare the calculation procedures for realistic operating conditions a set of theoretical vehicle loads was generated using a vehicle simulation program (Cole and Cebon 1988). Two simple linear vehicle models were used. The 'quarter-car' (figure 5) represents soft steel or air suspensions with a main dynamic component at 1.9Hz. The 'walking-beam' model (figure 6) has dominant frequencies at 2.8 and 9Hz. Both models simulated driving over a 'poor' random road profile, as defined in (Anon 1972). This profile was used in order to excite large dynamic loads that would yield worst-case differences between the convolution and influence function calculations. The impulse responses and influence functions were derived from measurements (Hardy and Cebon 1989). The damage inflicted along the road was calculated using the usual ε-N fatigue law with an exponent of 5, the 'rainflow' method of cycle counting and Miner's hypothesis (Hardy and Cebon 1992).

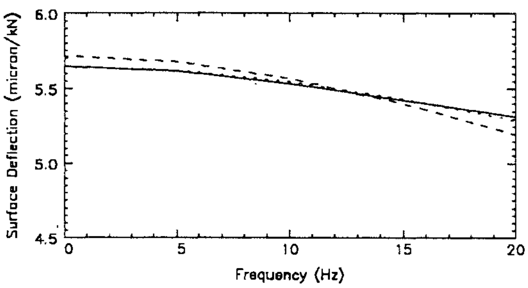
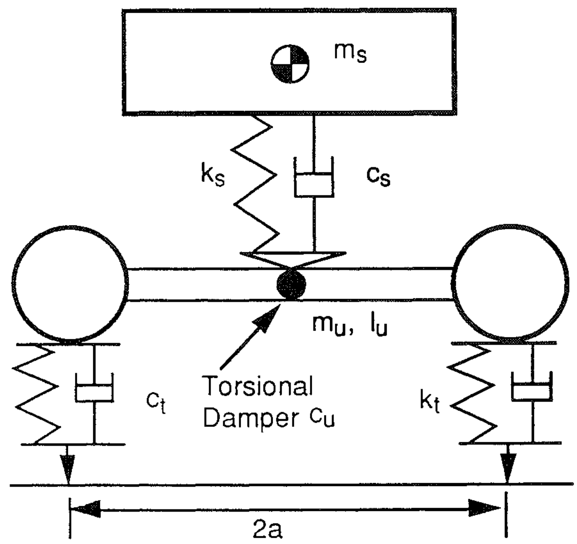



Figure 4 Frequency Response Functions.
Layered Half-Space ————,
Plate ·········
Beam – – – – ·




$m_s$ = 3900kg $k_s$ = 1MN/m $c_s$ = 15kNs/m
$m_u$ = 1100kg $k_t$ = 1.75MN/m $c_u$ = 1.5kNs/rad
$I_u$ = 465kgm$^2$ $c_t$ = 2kNs/m a = 0.65m

Figure 6 Walking Beam Vehicle Model.

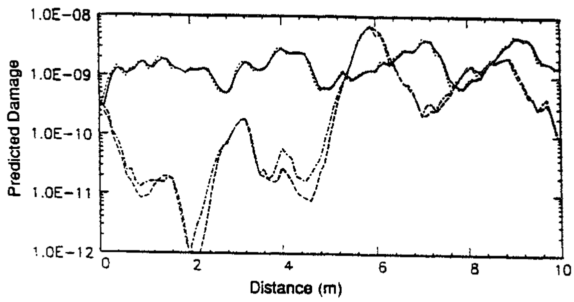



Figure 7 Predicted Damage at Points along a Road.
Quarter-car: Convolution ————,
Influence Function ............,
Walking Beam: Convolution — — — — -,
Influence Function -.-.-.-.-.-

Figure 7 shows examples of the variation of damage with distance calculated in this way. The differences between the two calculation procedures are evident in the predicted damage.

The *95th percentile* damage can be used to compare the level of damage suffered at the 5% of worst points along the road at different speeds (Cebon 1988a). The variation of the *95th percentile* damage with speed for both vehicle models and both calculation procedures is shown in figure 8. In each case the graphs are normalised by the damage incurred at 'creep speed'. It is worth noting that each convolution calculation took approximately thirty times longer than the equivalent influence function calculation. Normalised 95th percentile damage levels of 4–6 are more typical (Cebon and Winkler 1990).

It is apparent that the damage predicted by the two road response calculation procedures is virtually identical for both vehicle models. The very large damage levels for the walking-beam do not occur in practice because (i) heavy vehicles would not drive along such rough roads at high speeds, and (ii) in practice the tyres would lose contact with the road surface for the simulated conditions, but this did not occur for the linearised vehicle models.

The influence function simplification has been checked on one test road and there may be roads for which it is not valid. The road profile input to the vehicles was, however, much more uneven than typical trunk roads causing very large dynamic forces. It is thought that the worst-case results presented here obviate the need to perform a large parametric study. Verification on several more roads is thought desirable.

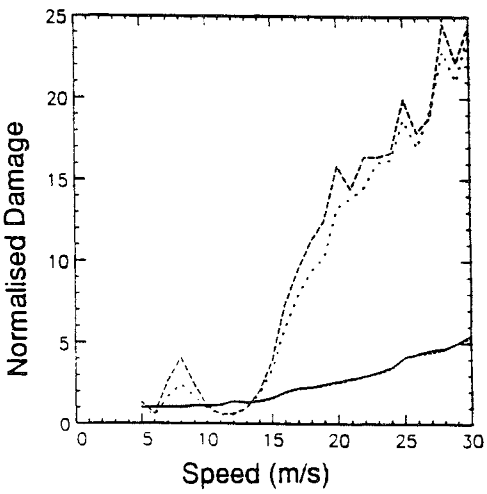



Figure 8 Normalised 95th Percentile Damage.
Quarter-car: Convolution ————,
Influence Function ............,
Walking Beam: Convolution — — — — -,
Influence Function -.-.-.-.-.-

## 6 CONCLUSIONS

(i) It is not necessary to model a complete road structure in order to simulate the dynamic response of the asphalt surface layers. The precise detail of the model required depends on the information that is required from it. In particular, it has been found that the strains in the surface layers are independent of the depth of the subgrade.

(ii) The influence function calculation is sufficient to model road responses to dynamic wheel loads as the errors introduced are very small. This does *not* mean that the influence function method is suitable to simulate responses to stationary dynamic loads such as the Falling Weight Deflectometer.

## 7 REFERENCES


Anon. "Pavement design: A guide to the structural design of road pavements." NAASRA, Sydney, Rept No. MEC-11, ISNBN 0 85588 209 3.

Anon. (1972). "Proposals for generalised road inputs to vehicles." Rept No. ISO/TC 108/WGp draft No.3e.

Cebon, D. (1987). "Assessment of the dynamic wheel forces generated by heavy road vehicles." ARRB/FORS Symposium on Heavy Vehicle Suspension Characteristics. 143-162.

Cebon, D. (1988a). "Theoretical road damage due to dynamic tyre forces of heavy vehicles. Part 1: Dynamic analysis of vehicles and road surfaces." *Proc I.Mech.E.* 202(C2): 103-108.

Cebon, D. (1988b). "Theoretical road damage due to dynamic tyre forces of heavy vehicles. Part 2:

Simulated damage caused by a tandem-axle vehicle." *Proc. I.Mech.E.* **202**(C2): 109-117.

Cebon, D. and C. B. Winkler. (1990). "A study of road damage due to dynamic wheel loads using a load measuring mat." Final report, Volume 1, Strategic Highway Research Program, Rept No. UMTRI-90-13.

Cole, D. J. and D. Cebon. (1988). "Simulation and measurement of vehicle response to road roughness." *Proc. Inst. Acoustics.* **10**(2): 477–484.

Fryba, L. (1972). Vibration of solids and structures under moving loads. Groningen, Netherlands, Noordhoff International Publishing.

Hardy, M. S. A. (1990). "The response of flexible pavements to dynamic tyre forces."

Hardy, M. S. A. and D. Cebon. (1989). "Measurement and analysis of the dynamic response of flexible pavements." Proc. 2nd Int. Symposium on heavy vehicle weights and dimensions.

Hardy, M. S. A. and D. Cebon. (1992). "The Effects of Dynamic Axle Loads on the Response and Life of Flexible Pavements." 7th International Conference on Asphalt Pavements.

Kausel, E. and R. Peek. (1982). "Dynamic Loads in the Interior of a Layered Stratum: An Explicit Solution." *Bulletin of the Seismological Soc. of America.* 72(5): 1459–1481.

Kausel, E. and J. M. Roësset. (1981). "Stiffness Matrices for Layered Soils." *Bull.Seism. Soc. America.* **71**: 1743–1761.

Newland, D. E. (1985). Random Vibrations and Spectral Analysis, 2nd Edition. Longman.

Semenov, V. A. (1976). "Stress–strain state of a composite structure consisting of a stack of elastic plates on a multilayer granular base." *Soviet Applied Mechanics.* **12**(3): 318–321.